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Abstract
We present a new derivation of the spectral gap of the totally asymmetric exclusion process on a half-filled ring of size L by using the Bethe ansatz. We show that, in the large-L limit, the Bethe equations reduce to a simple transcendental equation involving the polylogarithm, a classical special function. By solving that equation, the gap and the dynamical exponent are readily obtained. Our method can be extended to a system with an arbitrary density of particles.

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1. Introduction

The asymmetric simple exclusion process (ASEP) is a model of driven diffusive particles on a lattice with hard-core exclusion (for a general review, see Spohn (1991)). The ASEP appears as a minimal building block in a large variety of models for hopping conductivity (Richards 1977), polymer reptation (Widom *et al* 1991), traffic flow (Schreckenberg and Wolf 1998) or surface growth (Krug 1997). In particular, the ASEP in one dimension is a discrete version of the Kardar–Parisi–Zhang (KPZ) equation (Halpin-Healy and Zhang 1995). In biophysics, the ASEP has been used to describe the diffusion of macromolecules through narrow vessels (Levitt 1973) and the kinetics of protein synthesis on RNA (MacDonald and Gibbs 1969); more recently, a mapping between sequence-alignment and the exclusion process has been proposed (Bundschuh 2002). From a theoretical point of view, the ASEP plays the role of a paradigm in non-equilibrium statistical mechanics: it displays a variety of features such as boundary induced phase transitions (Krug 1991), spontaneous symmetry breaking in one dimension (Evans *et al* 1995) and dynamical phase separation (Evans *et al* 1998).

Exact results for the ASEP in one dimension have been derived using two complementary approaches (for a review, see Derrida (1998)). The matrix ansatz (Derrida *et al* 1993) allows us to calculate steady state properties such as invariant measures (Speer 1993), current

fluctuations in the stationary state and large deviation functionals (Derrida *et al* 2003). The Bethe ansatz (Dhar 1987) provides spectral information about the evolution (Markov) operator (Gwa and Spohn 1992, Schütz 1993, Kim 1995) which can be used to derive large deviation functions (Derrida and Lebowitz 1998, Derrida and Appert 1999, Derrida and Evans 1999). The exact relation between these two techniques is still a matter of investigation (Alcaraz *et al* 1994, Stinchcombe and Schütz 1995, Alcaraz and Lazo 2004).

The relaxation time to the stationary state for the ASEP on a lattice of size L scales typically as a power law, L^z, z being the ASEP dynamical exponent. The calculation of z for a one-dimensional system by Bethe ansatz is an important exact result that was first announced by Dhar, who found $z = 3/2$ (Dhar 1987). The spectral gap (and thus z) was subsequently calculated for the half-filling case by Gwa and Spohn (1992) and for an arbitrary density by Kim (1995) who mapped the ASEP onto a non-Hermitian XXZ Heisenberg spin chain. The one-dimensional ASEP belongs to the KPZ universality class and therefore the KPZ dynamical exponent in one dimension is equal to $3/2$; this result was previously deduced from Galilean invariance and renormalization group arguments (for a review, see Krug (1997)).

In this work, we present a new method of calculating the spectral gap of the totally asymmetric exclusion process (TASEP) starting from the Bethe ansatz equations. Our method, based on an analytic continuation formula, circumvents the technical difficulties involved in the derivation of Gwa and Spohn (1992) and renders the calculation of the ASEP dynamical exponent much more concise and transparent. Besides, our technique can be extended to the arbitrary density case and allows us to calculate the spectral gap for the asymmetric exclusion process with a tagged particle.

The outline of this paper is as follows. In section 2, we recall the definition of the TASEP, present the Bethe ansatz equations without deriving them and summarize their analysis. In section 3, we present our original calculation of the spectral gap in the half-filling case. Concluding remarks and generalizations of our method are given in section 4.

2. Bethe ansatz for the TASEP

2.1. The TASEP model

We consider the totally asymmetric simple exclusion process on a periodic one-dimensional lattice with L sites (sites i and $L + i$ are identical). In this model, the total number n of particles is conserved. Each lattice site i ($1 \leqslant i \leqslant L$) is either occupied by one particle or is empty (*exclusion rule*). Stochastic dynamical rules govern the evolution of the system: a particle on a site i at time t jumps, in the interval between times t and $t + \mathrm{d}t$, with a probability $\mathrm{d}t$ to the neighbouring site $i + 1$ if this site is empty. The total number of configurations for n particles on a ring with L sites is given by $\Omega = L!/[n!(L - n)!]$. In the stationary state, all configurations have the same probability $1/\Omega$ (Derrida 1998).

A configuration can be characterized by the positions of the n particles on the ring, $(x_1, x_2, \ldots, x_n)$ with $1 \leqslant x_1 < x_2 < \cdots < x_n \leqslant L$. We call $\psi_t(x_1, \ldots, x_n)$ the probability of this configuration at time t; the probability distribution ψ_t of the system at time t is thus a Ω-dimensional vector. As the ASEP is a continuous-time Markov (i.e., memoryless) process, the time evolution of ψ_t is determined by the master equation

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t} = M\psi_t \tag{1}$$

where the transition rate $\Omega \times \Omega$ matrix M is the Markov matrix. A right eigenvector ψ is associated with the eigenvalue E of M if

$$M\psi = E\psi. \tag{2}$$

The Markov matrix M is a real non-symmetric matrix and, therefore, its eigenvalues (and eigenvectors) are either real numbers or complex conjugate pairs. The spectrum of M contains the eigenvalue $E = 0$ and the associated right eigenvector is the stationary state $\psi(x_1, \ldots, x_n) = 1/\Omega$. Because the dynamics is ergodic (i.e., M is an irreducible and aperiodic Markov matrix), the Perron–Frobenius theorem implies that all eigenvalues E except 0 have a strictly negative real part; the relaxation time of the corresponding eigenmode is $\tau = -1/\text{Re}(E)$. (The imaginary part of E gives rise to oscillatory behaviour).

In this paper, we shall calculate the gap E_1, i.e., the non-zero eigenvalue of M with largest real part. The eigenmode associated with E_1 has thus the longest relaxation time that scales as L^z, z being the dynamical exponent.

2.2. The Bethe equations

Writing M explicitly, the eigenvalue equation (2) becomes

$$E\psi(x_1, \ldots, x_n) = \sum_i [\psi(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n) - \psi(x_1, \ldots, x_n)] \qquad (3)$$

where the sum runs over the indices i such that $x_{i-1} < x_i - 1$, i.e., such that the corresponding jump is allowed. The *Bethe ansatz* assumes that the eigenvectors ψ can be written in the form

$$\psi(x_1, \ldots, x_n) = \sum_{\sigma \in \Sigma_n} A_\sigma z_{\sigma(1)}^{x_1} z_{\sigma(2)}^{x_2} \cdots z_{\sigma(n)}^{x_n} \qquad (4)$$

where Σ_n is the group of the $n!$ permutations of n indices. The coefficients $\{A_\sigma\}$ and the fugacities $\{z_1, \ldots, z_n\}$ are complex numbers to be determined. The eigenvalue E associated with an eigenvector of the form (4) is given by

$$E = -n + \sum_{i=1}^{n} 1/z_i. \qquad (5)$$

Using matching conditions at the boundary surfaces $x_{i-1} = x_i - 1$ and the periodicity of the lattice, it can be shown that a vector ψ of the type (4) is an eigenvector of M if the fugacities $\{z_1, \ldots, z_n\}$ satisfy the *Bethe equations*

$$(z_i - 1)^n z_i^{-L} = -\prod_{j=1}^{n}(1 - z_j) \qquad \text{for} \quad i = 1, \ldots, n. \qquad (6)$$

The procedure for deriving these equations has been thoroughly explained in Halpin-Healy and Zhang (1995) and Derrida (1998).

The obvious solution $z_1 = \cdots = z_n = 1$ provides the stationary distribution with eigenvalue 0. More generally, given a solution $\{z_1, \ldots, z_n\}$ of equation (6), the corresponding eigenvalue E is obtained from equation (5); moreover, the coefficients $\{A_\sigma\}$ and thus the eigenvector ψ are uniquely determined. In order to have a complete basis of eigenvectors, Ω independent solutions of the Bethe equations (6) are needed.

Following Gwa and Spohn (1992), we introduce $Z_i = 2/z_i - 1$. Equations (5) and (6) then become, respectively,

$$2E = -n + \sum_{j=1}^{n} Z_j \qquad (7)$$

and

$$(1 - Z_i)^n(1 + Z_i)^{L-n} = -2^L \prod_{j=1}^{n} \frac{Z_j - 1}{Z_j + 1} \qquad \text{for} \quad i = 1, \ldots, n. \qquad (8)$$

We note that the left-hand side of equation (8) is a polynomial in Z_i whereas the right-hand side (rhs) is independent of the index i.

The analysis of the Bethe equations is simplified if only half-filled models are considered, that is, if $L = 2n$ (Gwa and Spohn 1992). Equation (8) then reduces to

$$\left(1 - Z_i^2\right)^n = -4^n \prod_{j=1}^{n} \frac{Z_j - 1}{Z_j + 1} \qquad \text{for} \quad i = 1, \ldots, n. \tag{9}$$

The half-filling restriction does not affect the physical behaviour of the ASEP: in the large-L limit, models with arbitrary density, $\rho = n/L \in {]}0, 1{[}$, belong to the same universality class. However, systems with vanishingly small density of particles ($\rho \to 0$) or holes ($\rho \to 1$) exhibit different behaviour and will not be discussed here.

2.3. Analysis of Bethe equations

Taking advantage of the fact that the rhs of equation (9) is independent of the index i, the Bethe equations can be reformulated as follows. Consider the polynomial equation

$$(1 - Z^2)^n = Y \tag{10}$$

where Y is a given complex number. Writing

$$Y = -e^{u\pi} \tag{11}$$

u being a complex number with $-1 \leqslant \text{Im}(u) < 1$, we obtain the nth roots of Y

$$y_m = e^{(u+i)\pi/n} \, e^{(m-1)2i\pi/n} \qquad \text{for} \quad m = 1, \ldots, n. \tag{12}$$

The y_m are evenly spaced on a circle of centre 0 and radius $|Y|^{1/n}$ and are labelled counterclockwise $0 \leqslant \text{Arg}(y_1) < \text{Arg}(y_2) < \cdots < \text{Arg}(y_n) < 2\pi$. Thus, the $2n$ solutions $(Z_1, \ldots, Z_{2n})$ of equation (10) are

$$Z_m = (1 - y_m)^{1/2}; \qquad Z_{m+n} = -Z_m \qquad \text{with} \quad m = 1, \ldots, n. \tag{13}$$

The branch cut of the function $z^{1/2}$ is, as usual, the real semi-axis $(-\infty, 0]$, i.e., for $m = 1, \ldots, n$, the argument of Z_m belongs to $[-\pi/2, \pi/2[$. We explain in appendix B that each Z_m is an analytic function of Y in the complex plane with a branch cut along $[0, +\infty)$ and that the locus of the Z_m is a remarkable curve called a Cassini oval.

We now choose n different roots $(Z_{c(j)})_{j=1,n}$ among $(Z_1, \ldots, Z_{2n})$, such that the choice function $c : \{1, \ldots, n\} \to \{1, \ldots, 2n\}$ satisfies

$$1 \leqslant c(1) < \cdots < c(n) \leqslant 2n. \tag{14}$$

There are precisely Ω such choice functions, $\Omega = (2n)!/n!^2$ being the size of the Markov matrix. Finally, we define

$$A_c(Y) = -4^n \prod_{j=1}^{n} \frac{Z_{c(j)} - 1}{Z_{c(j)} + 1} \tag{15}$$

where A_c is a function of Y and of the choice function c. The Bethe equation (9) is equivalent to the *self-consistency* equation

$$A_c(Y) = Y. \tag{16}$$

Given the choice function c and a root Y of this equation, the $Z_{c(j)}$ are determined from equation (10) and the corresponding eigenvalue E_c is obtained from equation (7).

For small values of n, the above described procedure allows us to compute numerical solutions of the Bethe equations. From our numerical observations, we conjecture that for each choice function c (among the Ω possible choice functions), the self-consistency equation (16) has a unique solution Y that yields one eigenvector ψ_c and one eigenvalue E_c. This suggests that the Bethe equations yield a complete basis of eigenvectors for the ASEP.

Let us first consider the choice function $c(j) = j$, i.e., the selected Z_j are the n solutions of equation (10) with largest real parts. As this choice plays an important role in the following analysis, we define

$$A_0(Y) = -4^n \prod_{j=1}^n \frac{Z_j - 1}{Z_j + 1} \tag{17}$$

$$2E_0 = -n + \sum_{j=1}^n Z_j. \tag{18}$$

We emphasize that E_0 is an implicit function of Y. The equation $A_0(Y) = Y$ has the solution $Y = 0$ that yields $Z_j = 1$ for all j and provides the stationary distribution (or ground state) with eigenvalue 0.

Numerical observations (Gwa and Spohn 1992) indicate that the first excited eigenvalue E_1 corresponds to the choice $c(j) = j$ for $j = 1, \ldots, n - 1$ and $c(n) = n + 1$; i.e., the first excited state is obtained from the ground state by the *excitation* $(n \to n + 1)$. Writing $A_1(Y)$ and E_1 for the functions A_c and E_c corresponding to this choice function, we have, from equations (15), (17) and (18),

$$A_1(Y) = A_0(Y) \left(\frac{Z_1 - 1}{Z_1 + 1} \frac{Z_n - 1}{Z_n + 1} \right)^{-1} \tag{19}$$

$$2E_1 = 2E_0 - (Z_1 + Z_n) \tag{20}$$

where we have used $Z_{n+1} = -Z_1$. The excitation $(1 \to 2n)$, that is, $c(j) = j + 1$ for $j = 1, \ldots, n - 1$ and $c(n) = 2n$, also leads to equations (19) and (20) and thus to the same eigenvalue E_1. The first excited state has therefore a degeneracy of order 2.

Consequently, in order to find the expression for the gap E_1, we must solve the self-consistency equation

$$A_1(Y) = Y \tag{21}$$

then calculate the Z_j for $j = 1, \ldots, n$, and finally deduce E_1 from equation (20).

In the above discussion, we closely followed Gwa and Spohn (1992) to present the Bethe ansatz equations for the TASEP. We shall now solve these equations and calculate the gap by a radically different and simpler method.

3. Calculation of the gap

Let us define $F(Y)$ as

$$A_0(Y) = Y \exp(F(Y)). \tag{22}$$

In appendix A, we derive the following identities, valid for $|Y| \leqslant 1$,

$$F(Y) = \sum_{k=1}^{\infty} \frac{w_{kn}}{k} Y^k \tag{23}$$

$$-4E_0 = \sum_{k=1}^{\infty} \frac{w_{kn-1}}{k} Y^k \tag{24}$$

where the w_k are given by

$$w_k = \frac{(2k - 1)!!}{(2k)!!} = \frac{(2k)!}{(k!2^k)^2}. \tag{25}$$

From the Stirling formula, the leading order behaviour of w_k for $k \to \infty$, is given by

$$w_k \sim \frac{1}{\sqrt{\pi k}}. \tag{26}$$

From the power series (23) and (24) we deduce that $A_0(Y)$ and E_0 are analytic functions of the complex variable Y inside the unit circle. This property is not obvious *a priori*: the functions $A_0(Y)$ and E_0, defined in equations (17) and (18), respectively, depend implicitly on Y via the Z_m that involve a branch cut along $[0, +\infty)$. Indeed, for a generic choice function $c(j)$, $A_c(Y)$ and E_c are analytic only in the complex Y plane with a cut along $[0, +\infty)$ and, therefore, are not analytic in the neighbourhood of $Y = 0$. This special property of $A_0(Y)$ and E_0 is obtained by an explicit calculation in appendix A and from geometrical considerations in appendix B.

Using equations (19), (22) and (23), the self-consistency equation (21), that determines the gap, reduces to

$$\sum_{k=1}^{\infty} \frac{w_{kn}}{k} Y^k = \ln \frac{1 - Z_1}{1 + Z_1} + \ln \frac{1 - Z_n}{1 + Z_n}. \tag{27}$$

From equations (20) and (24), we have

$$-4E_1 = \sum_{k=1}^{\infty} \frac{w_{kn-1}}{k} Y^k + 2Z_1 + 2Z_n. \tag{28}$$

Combining equations (27) and (28), we obtain

$$-4E_1 = \sum_{k=1}^{\infty} \frac{w_{kn}}{k(2kn - 1)} Y^k + \left(2Z_1 + \ln \frac{1 - Z_1}{1 + Z_1} \right) + \left(2Z_n + \ln \frac{1 - Z_n}{1 + Z_n} \right) \tag{29}$$

where we have used $(2kn)w_{kn} = (2kn - 1)w_{kn-1}$.

Thus, to find the gap for a half-filled system with n particles, we must solve equation (27) for Y and substitute the result in equation (29). We emphasize that the power series in these equations represent, inside the unit disc, analytic functions of Y that are defined in the whole complex plane with a cut along $[1, +\infty)$.

We now consider the thermodynamic limit, $n \to \infty$. We obtain, at leading order, from equation (26)

$$F(Y) = \sum_{k=1}^{\infty} \frac{w_{kn}}{k} Y^k \quad \to \quad \frac{1}{\sqrt{\pi n}} \mathrm{Li}_{3/2}(Y) \tag{30}$$

$$\sum_{k=1}^{\infty} \frac{w_{kn}}{k(2kn - 1)} Y^k \quad \to \quad \frac{1}{2\sqrt{\pi n^3}} \mathrm{Li}_{5/2}(Y) \tag{31}$$

where we have used the *polylogarithm* function of index s

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \tag{32}$$

By virtue of the integral representation

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} \, \mathrm{d}t}{\mathrm{e}^t - z} \tag{33}$$

the function Li_s can be extended by analytic continuation to the whole complex plane with a branch cut along the real semi-axis $[1, +\infty)$. In the large-n limit, we deduce from equations (11), (12) and (13) that

$$Z_1 = (1 - y_1)^{1/2} = \sqrt{\frac{\pi}{n}}(-u - i)^{1/2} + O\left(\frac{1}{n^{3/2}}\right) \tag{34}$$

$$Z_n = (1 - y_n)^{1/2} = \sqrt{\frac{\pi}{n}}(-u + i)^{1/2} + O\left(\frac{1}{n^{3/2}}\right) \tag{35}$$

where we have supposed that $Y = -e^{u\pi}$ remains finite when $n \to \infty$. Using these expressions and the expansion $\ln \frac{1-Z}{1+Z} = -2Z - \frac{2}{3}Z^3 + O(Z^5)$, equation (27) reduces to

$$\text{Li}_{3/2}(-e^{u\pi}) = -2\pi[(-u + i)^{1/2} + (-u - i)^{1/2}] \tag{36}$$

and the gap (29), at the leading order, is given by

$$E_1 = \frac{1}{n^{3/2}}\left\{\frac{-1}{8\sqrt{\pi}}\text{Li}_{5/2}(-e^{u\pi}) + \frac{\pi^{3/2}}{6}[(-u + i)^{3/2} + (-u - i)^{3/2}]\right\}. \tag{37}$$

(Note that the rhs of equation (36) and of equation (37) are real when u is real.) With the help of the Maple software, we find a unique solution of equation (36) in the strip $-1 \leqslant \text{Im}(u) < 1$ that is real and is given by

$$u = 1.119\,068\,802\,804\,474\cdots \tag{38}$$

Inserting this value of u in equation (37) yields the large-n (or large-L) behaviour of the gap

$$E_1 = -\frac{2.301\,345\,960\,455\,050\cdots}{n^{3/2}} = -\frac{6.509\,189\,337\,976\,136\cdots}{L^{3/2}}. \tag{39}$$

This is precisely the result obtained by Gwa and Spohn (1992). This gap scales as $L^{-3/2}$ and is real for the TASEP in the half-filling case.

4. Summary and discussion

In this work, we have calculated the gap of the TASEP in the limit of a large size system by using the Bethe ansatz. We first take the large-n limit of the Bethe equations inside the unit circle, then perform the analytic continuation of these equations in the whole complex plane with a branch cut along $[1, +\infty)$ and finally solve them. Gwa and Spohn (1992), in contrast, first represent the analytic continuation of the Bethe equations for a fixed value of n as an n-dependent complex integral (thanks to the Euler–Maclaurin formula) and then extract the gap from the large-n limit of this integral representation which is rather a delicate operation. We have shown here that the derivation of the TASEP gap is greatly simplified by performing the calculations in the reverse order, that is taking the large-n limit first and the analytic continuation afterwards.

We do not claim that it is always true that large-n limit and analytic continuation are commuting operations. If the solution Y of equation (21), in the large-n limit, diverges to ∞ or approaches asymptotically the branch cut, reversing the order of operations may not be possible. Fortunately, in our problem, Y remains a bounded negative real number when $n \to \infty$.

We have applied our method to several other problems but, for the sake of conciseness, we simply list these additional calculations leaving the details to a future publication. The gap for the totally asymmetric exclusion process can also be calculated for an arbitrary density $\rho = n/L$: the L roots Z_i of the Bethe equation (8) now belong to a deformed Cassini curve; for small Y, this curve consists of two ovals, the roots $(Z_1, \ldots, Z_n)$ lying on the right oval and $(Z_{n+1}, \ldots, Z_L)$ on the left oval. Consequently, the function $A_0(Y)$, defined in equation (17), remains an analytic function of Y and its power series expansion is explicitly obtained by

using contour integration in the complex plane. Taking the thermodynamic limit, we find, in agreement with Kim (1995), that the gap $E_1(\rho)$ is given by

$$E_1(\rho) = 2\sqrt{\rho(1-\rho)}\,E_1(\rho = 1/2) \pm \frac{2i\pi}{L}(2\rho - 1) \tag{40}$$

where $E_1(\rho = 1/2)$ is the gap for the half-filling case given in equation (39); we note that, for $\rho \neq 1/2$, the gap has a non-zero imaginary part.

Our technique also provides the subleading corrections to the gap and allows us to calculate the eigenvalue of the highest excited state, of a finite excitation above the ground state, or below the highest excited state. Besides, we have studied generalizations of the ASEP by introducing a tagged particle that has the same dynamics as the other particles: the gap then scales as $L^{-5/2}$.

Finally, we emphasize that formula (33) already appeared in the work of Derrida and Appert (1999): indeed, equations (27) and (28) are similar to those used in their calculation of large deviation functions of the ASEP by the Bethe ansatz.

Acknowledgments

Appendix A. Derivation of equations (23) and (24)

The numbers w_k defined in equation (25) are the coefficients of the Taylor series

$$\frac{1}{\sqrt{1-x}} = \sum_{k=0}^{\infty} w_k x^k. \tag{A.1}$$

By integration, we find

$$\sqrt{1-x} = 1 - \frac{1}{2}\sum_{k=1}^{\infty} \frac{w_{k-1}}{k}x^k. \tag{A.2}$$

Recalling that $Z_m = \sqrt{1 - y_m}$ (see equations (12) and (13)), we obtain

$$\sum_{m=1}^{n} Z_m = n - \frac{1}{2}\sum_{k=1}^{\infty} \frac{w_{k-1}}{k}\sum_{m=1}^{n} y_m^k. \tag{A.3}$$

The fact that the y_m are the nth roots of Y leads to the following relation:

$$\sum_{m=1}^{n} y_m^k = \begin{cases} nY^{k/n} & \text{if } k \text{ is a multiple of } n \\ 0 & \text{otherwise.} \end{cases} \tag{A.4}$$

Inserting this relation in equation (A.3), we obtain

$$\sum_{m=1}^{n} Z_m = n - \frac{1}{2}\sum_{k=1}^{\infty} \frac{w_{nk-1}}{k}Y^k. \tag{A.5}$$

We thus find, thanks to the crucial identity (A.4), that $\sum_{m=1}^{n} Z_m$ is analytic in Y inside the unit circle. Finally, substituting equation (A.5) into equation (18), we obtain equation (24).

The derivation of equation (23) follows similar steps. We first note that the Taylor expansion of the function

$$f(x) = \ln\left(\frac{4}{x}\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right) \tag{A.6}$$

is given by

$$f(x) = \sum_{k=1}^{\infty} \frac{w_k}{k} x^k. \tag{A.7}$$

(This follows from $f(0) = 0$ and $f'(x) = \frac{1}{x}(\frac{1}{\sqrt{1-x}} - 1)$). Using equation (17) and the identity $\prod_{m=1}^{n}(-y_m) = -Y$, we deduce that

$$A_0(Y) = Y \prod_{m=1}^{n} \frac{4}{y_m} \frac{1 - Z_m}{1 + Z_m}. \tag{A.8}$$

From equation (22) and equations (A.6)–(A.8), we obtain

$$F(Y) = \sum_{m=1}^{n} f(y_m) = \sum_{k=1}^{\infty} \frac{w_k}{k} \sum_{m=1}^{n} y_m^k. \tag{A.9}$$

This equation reduces to equation (23) by virtue of the identity (A.4).

Appendix B. Roots of the Bethe equations and Cassini ovals

The polynomial equation

$$(1 - Z^2)^n = Y \tag{B.1}$$

where Y is a fixed complex number, has $2n$ solutions, $(Z_1, \ldots, Z_{2n})$. The purpose of this appendix is to explain how these solutions can be labelled in a coherent way so that each root $Z_m(Y)$ is an analytic function of Y.

We first note that y_m, defined in equation (12), is an analytic function of Y in the complex plane with a branch cut along the real semi-axis $[0, +\infty)$. Nevertheless when Y crosses $[0, +\infty)$, the functions $y_1, y_2, \ldots, y_n$, are the analytic continuations (above the axis) of respectively $y_n, y_1, \ldots, y_{n-1}$: thus the existence of the branch cut along $[0, +\infty)$ is due to the labelling of the roots.

The complex numbers $(Z_1, \ldots, Z_{2n})$ belong to the curve defined by

$$|Z - 1||Z + 1| = r \qquad \text{with} \quad r = |Y|^{1/n} \tag{B.2}$$

and called a *Cassini oval*. A Cassini oval is the conformal transformation of the circle of centre 1 and radius r by the function $z \to z^{1/2}$. Its shape depends on whether the point 0 is inside or outside the circle, i.e., whether $r < 1$ or not. When $r < 1$, the curve of equation (B.2) consists of two ovals around the points $Z = \pm 1$. The numbers $(Z_1, \ldots, Z_n)$ lie on the right oval and $(Z_{n+1}, \ldots, Z_{2n})$ on the left oval. For the marginal case, $r = 1$, the curve is the *lemniscate of Bernoulli*, with a multiple point at $Z = 0$. When $r > 1$, the Cassini oval is a single loop with a peanut shape (when $r \in]1, 2[$) or an oval shape ($r \geqslant 2$). See figure 1 where the cases $r = 0.9$, 1 and 1.1 are drawn. In the large-r limit, the oval tends to the circle of radius $\sqrt{r}$.

We now discuss the analyticity properties of $Z_m(Y)$: Z_m is an analytic function of y_m with a branch cut along $(-\infty, 0]$; this branch cut is compatible with that of $y_m(Y)$. Consequently Z_m is an analytic function of Y with a branch cut along $[0, +\infty)$. Moreover when Y crosses the real segment $[0, 1[$, the functions $Z_1(Y), Z_2(Y), \ldots, Z_n(Y)$ are the analytic continuations (above the axis) of the functions $Z_n(Y), Z_1(Y), \ldots, Z_{n-1}(Y)$ respectively. See figure 1. (A similar property is true for the functions $Z_{n+1}(Y), Z_{n+2}(Y), \ldots, Z_{2n}(Y)$ that belong to the left oval.) But when Y crosses $]1, +\infty)$, the functions $Z_1(Y), Z_2(Y), \ldots, Z_{2n}(Y)$ are the analytic continuations (above the axis) of the functions $Z_{2n}(Y), Z_1(Y), \ldots, Z_{2n-1}(Y)$
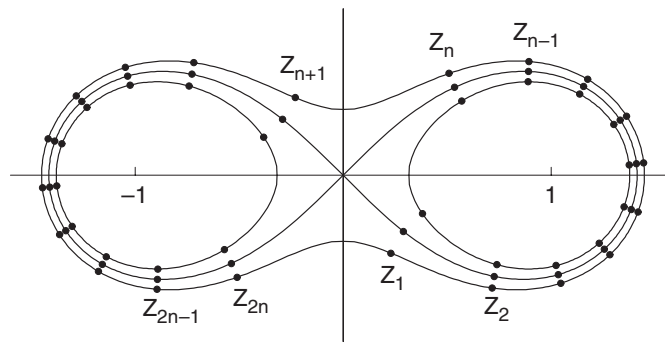
Figure 1. Labelling the roots Z_m of the equation $(1 - Z^2)^n = Y$. Here $Y = e^{i\phi} r^n$ with $n = 10$, $\phi = \pi/2$ and $r \in \{0.9, 1, 1.1\}$. The continuous curves are the corresponding Cassini ovals (see text). When r is fixed and ϕ goes from 0 to 2π, each Z_m slips counterclockwise along the Cassini ovals. Then, the jump $\phi = 2\pi \to 0$, i.e., Y crosses $[0, +\infty)$, consists of a global shift of the labels m around each continuous curve.

respectively. Consequently the branch cut of the function $A_0(Y)$ defined in equation (17) is $[1, +\infty)$ and not $[0, +\infty)$.

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